

Darboux Transformation and Exact Solutions of the Integrable Heisenberg Ferromagnetic Equation with Self-Consistent Potentials

Z.S. Yersultanova, M. Zhassybayeva, K. Yesmakhanova, G. Nugmanova and
R. Myrzakulov*

*Eurasian International Center for Theoretical Physics and Department of General
& Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan*

Abstract

Integrable Heisenberg ferromagnetic equations are an important subclass of integrable systems. The M-XCIX equation is one of a generalizations of the Heisenberg ferromagnetic equation and are integrable. In this paper, the Darboux transformation of the M-XCIX equation is constructed. Using the DT, a 1-soliton solution of the M-XCIX equation is presented.

1 Introduction

During the past decades, there has been an increasing interest in the study of nonlinear models, especially, integrable nonlinear differential equations. Such integrable equations admit in particular soliton or soliton-like solutions. The study of the solitons and related solutions have become one of active areas of research in physics and mathematics. There are several methods to find soliton and other exact solutions of integrable equations, for instance, Hirota method, inverse scattering transformation, bilinear method, Darboux transformation and so on. Among these methods, the Darboux transformation (DT) is a efficient method to construct the exact solutions of integrable equations.

Among of integrable systems, the Heisenberg ferromagnetic equation (HFE) plays an important role in physics and mathematics [1]-[15]. In particular, it describes nonlinear dynamics of magnets. Also the HFE can reproduce some integrable classes of curves and surfaces in differential geometry [16]-[26]. There are several types integrable HFE. In this paper, we construct the DT for the HFE with self-consistent potentials [27]-[33]. Using the DT, we provide soliton solutions of the HFE with self-consistent potentials.

The paper is organized as follows. In section 2, the M-XCIX equation and its Lax representation are introduced. In section 3, we derived the DT of the M-XCIX equation. Using these Darboux transformations, one soliton solutions are derived in section 4. Section 5 is devoted to conclusion.

*Email: rmyrzakulov@gmail.com

2 The M-XCIX equation

The Myrzakulov-XCIX equation (M-XCIX equation) reads as [23]-[24] (about our notation and definitions, see e.g. [11]-[22])

$$iS_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (2.1)$$

$$iW_x + \omega[S, W] = 0, \quad (2.2)$$

where $S = S_i \sigma_i$, $W = W_i \sigma_i$, $S^2 = I$, $W^2 = b(t)I$, $b(t) = \text{const}(t)$, $I = \text{diag}(1, 1)$, $[A, B] = AB - BA$, ω is a real constant and σ_i are Pauli matrices. The M-XCIX equation is integrable by the IST. Its Lax representation can be written in the form

$$\Phi_x = U\Phi, \quad (2.3)$$

$$\Phi_t = V\Phi, \quad (2.4)$$

where the matrix operators U and V have the form

$$U = -i\lambda S, \quad (2.5)$$

$$V = \lambda^2 V_2 + \lambda V_1 + \left(\frac{i}{\lambda + \omega} - \frac{i}{\omega} \right) W. \quad (2.6)$$

Here

$$V_2 = -2iS, \quad V_1 = 0.5[S, S_x], \quad (2.7)$$

$$S = \begin{pmatrix} S_3 & S^+ \\ S^- & -S_3 \end{pmatrix}, \quad W = \begin{pmatrix} W_3 & W^+ \\ W^- & -W_3 \end{pmatrix}, \quad (2.8)$$

$$S^\pm = S_1 \pm iS_2, \quad W^\pm = W_1 \pm iW_2. \quad (2.9)$$

At last, we note that if $W = 0$ then the M-XCIX equation becomes the usual HFE

$$iS_t + \frac{1}{2}[S, S_{xx}] = 0. \quad (2.10)$$

3 Darboux transformation

3.1 One-fold DT

In this section, we construct the DT for the M-XCIX equation. To do this, let us consider the following transformation of solutions of the equations (2.3)-(2.4)

$$\Phi' = L\Phi, \quad (3.1)$$

where

$$L = \lambda N - I \quad (3.2)$$

and

$$N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}. \quad (3.3)$$

We require that Φ' satisfies the same Lax representation as (2.3)-(2.4) so that

$$\Phi'_x = U'\Phi', \quad (3.4)$$

$$\Phi'_t = V'\Phi', \quad (3.5)$$

where $U' - V'$ depend on S' and W' as $U - V$ on S and W . The matrix L obeys the following equations

$$L_x + LU = U'L, \quad (3.6)$$

$$L_t + LV = V'L. \quad (3.7)$$

These equations yield the following equations for N

$$N_x = iS' - iS, \quad (3.8)$$

$$N_t = -S'S'_x - \frac{i}{\omega}W'N + \frac{i}{\omega}NW + SS_x \quad (3.9)$$

and

$$S' = NSN^{-1}. \quad (3.10)$$

$$W' = (I + \omega N)W(I + \omega N)^{-1}. \quad (3.11)$$

Also we have the following useful second form of the DT for S :

$$S' = S - iN_x. \quad (3.12)$$

3.1.1 One-fold DT in terms of the N matrix

We now ready to write the DT for the M-XCIX equation in the explicit form. It can be shown that the matrix N has the form

$$N = \begin{pmatrix} n_{11} & n_{12} \\ -n_{12}^* & n_{11}^* \end{pmatrix}, \quad (3.13)$$

so that

$$N^{-1} = \frac{1}{n} \begin{pmatrix} n_{11}^* & -n_{12} \\ n_{12}^* & n_{11} \end{pmatrix} \quad (3.14)$$

and

$$I + \omega N = -L_{|\lambda=-\omega} = \begin{pmatrix} \omega n_{11} + 1 & \omega n_{12} \\ -\omega n_{12}^* & \omega n_{11}^* + 1 \end{pmatrix}, \quad (I + \omega N)^{-1} = \frac{1}{\square} \begin{pmatrix} \omega n_{11}^* + 1 & -\omega n_{12} \\ \omega n_{12}^* & \omega n_{11} + 1 \end{pmatrix}. \quad (3.15)$$

Here

$$n = \det N = |n_{11}|^2 + |n_{12}|^2, \quad \square = \det(I + \omega N) = \omega^2 |n_{11}|^2 + \omega(n_{11} + n_{11}^*) + 1 + \omega^2 |n_{12}|^2. \quad (3.16)$$

Finally we have

$$\begin{aligned} S' &= \frac{1}{n} \begin{pmatrix} S_3(|n_{11}|^2 - |n_{12}|^2) + S^- n_{11} n_{12}^* + S^+ n_{11}^* n_{12} & S^- n_{11}^2 - S^+ n_{12}^2 - 2S_3 n_{11} n_{12} \\ S^+ n_{11}^* - S^- n_{12}^* - 2S_3 n_{11}^* n_{12}^* & S_3(|n_{12}|^2 - |n_{11}|^2) - S^- n_{11} n_{12}^* - S^+ n_{11}^* n_{12} \end{pmatrix} \\ W' &= \frac{1}{\square} \begin{pmatrix} 1 + A_{11} & A_{12} \\ -A_{21} & 1 - A_{22} \end{pmatrix}, \end{aligned} \quad (3.17)$$

where

$$A_{11} = (\omega^2 |n_{11}|^2 + \omega(n_{11} + n_{11}^*) - |n_{12}|^2)W_3 + (\omega n_{11}^* + 1)n_{12}W^+ + (\omega n_{11} + 1)n_{12}^*W^- \quad (3.19)$$

$$A_{12} = -2\omega n_{11} n_{12} W_3 - 2n_{12} W_3 + \omega^2 n_{11}^2 W^- + 2\omega n_{11} W^- + W^- - n_{12}^2 W^+, \quad (3.20)$$

$$A_{21} = -2\omega n_{11}^* n_{12}^* W_3 - 2n_{12}^* W_3 + \omega^2 (n_{11}^*)^2 W^+ + 2\omega n_{11}^* W^+ + W^+ - (n_{12}^*)^2 W^-, \quad (3.21)$$

$$A_{22} = (\omega^2 |n_{11}|^2 + \omega(n_{11} + n_{11}^*) - |n_{12}|^2)W_3 + (\omega n_{11}^* + 1)n_{12}W^+ + (\omega n_{11} + 1)n_{12}^*W^- \quad (3.22)$$

At last, we give the another form of the solutions of S as:

$$S' = S - i \begin{pmatrix} n_{11x} & n_{12x} \\ -n_{12x}^* & n_{11x}^* \end{pmatrix}, \quad (3.23)$$

so that

$$S^{+'} = S^+ + i n_{12x}^*, \quad (3.24)$$

$$S^{-'} = S^- - i n_{12x}, \quad (3.25)$$

$$S'_3 = S_3 - i n_{11x}. \quad (3.26)$$

3.1.2 One-fold DT in terms of eigenfunctions

Let the column $(\psi_1, \psi_2)^T$ is the solution of Eqs.(2.3)-(2.4) with λ . Then the new column $(-\psi_2^*, \psi_1^*)^T$ is the solution of Eqs.(2.3)-(2.4) as λ^* . We now consider the following matrix solution

$$H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & \psi_1(\lambda_2; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_2(\lambda_2; t, x, y) \end{pmatrix}. \quad (3.27)$$

It satisfies the system:

$$H_x = -iSH\Lambda, \quad (3.28)$$

$$H_t = -2iSH\Lambda^2 + SS_xH\Lambda - \frac{i}{\omega}WH + WH\Sigma, \quad (3.29)$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \frac{i}{\lambda_1 + \omega} & 0 \\ 0 & \frac{i}{\lambda_2 + \omega} \end{pmatrix}, \quad (3.30)$$

$\det H \neq 0$ and λ_k are complex constants. We now assume that the matrix N can be written as:

$$N = H\Lambda^{-1}H^{-1}. \quad (3.31)$$

From these equations follow that N obeys the equations

$$N_x = iNSN^{-1} - iS, \quad (3.32)$$

$$N_t = SS_x - NSS_xN^{-1} - \frac{i}{\omega}(WN - NW) + WH\Sigma\Lambda^{-1}H^{-1} - NWH\Sigma H^{-1}, \quad (3.33)$$

which are equivalent to Eqs.(3.8)-(3.9) as we expected. In order to satisfy the constraints of S and W , the S and matrix solutions of the system (2.3)-(2.4) obey the condition

$$\Phi^\dagger = \Phi^{-1}, \quad S^\dagger = S, \quad (3.34)$$

which follow from the equations

$$\Phi_x^\dagger = i\lambda\Phi^\dagger S^\dagger, \quad (\Phi^{-1})_x = i\lambda\Phi^{-1}S^{-1}. \quad (3.35)$$

Here \dagger denote an Hermitian conjugate. After some calculations we came to the formulas

$$\lambda_2 = \lambda_1^*, \quad H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & -\psi_2^*(\lambda_1; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & \psi_1^*(\lambda_1; t, x, y) \end{pmatrix}, \quad (3.36)$$

$$H^{-1} = \frac{1}{\Delta} \begin{pmatrix} \psi_1^*(\lambda_1; t, x, y) & \psi_2^*(\lambda_1; t, x, y) \\ -\psi_2(\lambda_1; t, x, y) & \psi_1(\lambda_1; t, x, y) \end{pmatrix}, \quad (3.37)$$

where

$$\Delta = |\psi_1|^2 + |\psi_2|^2. \quad (3.38)$$

So finally for the matrix N we get the following expression

$$N = \frac{1}{\Delta} \begin{pmatrix} \lambda_1^{-1}|\psi_1|^2 + \lambda_2^{-1}|\psi_2|^2 & (\lambda_1^{-1} - \lambda_2^{-1})\psi_1\psi_2^* \\ (\lambda_1^{-1} - \lambda_2^{-1})\psi_1^*\psi_2 & \lambda_1^{-1}|\psi_2|^2 + \lambda_2^{-1}|\psi_1|^2 \end{pmatrix}. \quad (3.39)$$

Hence we can write the DT in terms of the eigenfunctions of the Lax representations (2.3)-(2.4) as

$$S^{+'} = S^+ + i \left(\frac{(\lambda_1^{-1} - \lambda_2^{-1})\psi_1^*\psi_2}{\Delta} \right)_x, \quad (3.40)$$

$$S^{-'} = S^- - i \left(\frac{(\lambda_1^{-1} - \lambda_2^{-1})\psi_1\psi_2^*}{\Delta} \right)_x, \quad (3.41)$$

$$S'_3 = S_3 - i \left(\frac{\lambda_1^{-1}|\psi_1|^2 + \lambda_2^{-1}|\psi_2|^2}{\Delta} \right)_x \quad (3.42)$$

and similarly for W .

Lastly let us unify our notations rewriting the 1-fold DT as:

$$\Phi^{[1]} = L_1 \Phi \quad (3.43)$$

where

$$L_1 = \lambda l_1^1 + l_1^0 = \lambda l_1^1 - I \quad (3.44)$$

and $l_1^0 = -I$. Then the 1-fold DT takes the form

$$S^{[1]} = l_1^1 S(l_1^1)^{-1}, \quad (3.45)$$

$$W^{[1]} = L_1|_{\lambda=-\omega} W L_1^{-1}|_{\lambda=-\omega} \quad (3.46)$$

or

$$S^{[1]} = L_{1\lambda} S(L_{1\lambda})^{-1}, \quad (3.47)$$

$$W^{[1]} = L_1|_{\lambda=-\omega} W L_1^{-1}|_{\lambda=-\omega}. \quad (3.48)$$

3.2 Two-fold DT

In this subsection we want give some main formulas of the 2-fold DT. We start from the transformation

$$\Phi^{[2]} = L_2 \Phi^{[1]} = (\lambda N_2 - I) \Phi^{[1]} = (\lambda N_2 - I)(\lambda N_1 - I) \Phi = (l_2^0 + \lambda l_2^1 + \lambda^2 l_2^2) \Phi \quad (3.49)$$

where

$$L_2 = l_2^0 + \lambda l_2^1 + \lambda^2 l_2^2. \quad (3.50)$$

Here

$$l_2^1 = -(N_1 + N_2), \quad l_2^2 = N_2 N_1, \quad l_2^0 = I \quad (3.51)$$

We have

$$L_{2x} + L_2 U = U^{[2]} L_2, \quad (3.52)$$

$$L_{2t} + L_2 V = V^{[2]} L_2. \quad (3.53)$$

Then Eq.(3.52) gives the coefficients of λ^i as:

$$\lambda^0 : l_{2x}^0 = 0 \quad (3.54)$$

$$\lambda^1 : l_{2x}^1 = i l_2^0 S - i S^{[2]} l_2^0, \quad (3.55)$$

$$\lambda^2 : l_{2x}^2 = i l_2^1 S - i S^{[2]} l_2^1. \quad (3.56)$$

$$\lambda^3 : 0 = i l_2^2 S - i S^{[2]} l_2^2. \quad (3.57)$$

Hence in particular we get

$$S^{[2]} = l_2^2 S (l_2^2)^{-1}. \quad (3.58)$$

We need in the following formulas:

$$\frac{\lambda}{\lambda + \omega} = 1 - \frac{\omega}{\lambda + \omega}, \quad \frac{\lambda^2}{\lambda + \omega} = \lambda - \omega + \frac{\omega^2}{\lambda + \omega}. \quad (3.59)$$

Then coefficients of λ^i of two sides of the equation (3.53) give us the following equations

$$\lambda^0 : l_{2t}^0 - \frac{i}{\omega} l_2^0 W + i l_2^1 W - i \omega l_2^2 W = -\frac{i}{\omega} W^{[2]} l_2^0 + i W^{[2]} l_2^1 - i \omega W^{[2]} l_2^2, \quad (3.60)$$

$$\lambda^1 : l_{2t}^1 + l_2^0 S S_x - \frac{i}{\omega} l_2^1 W + i l_2^2 W = S^{[2]} S_x^{[2]} l_2^0 - \frac{i}{\omega} W^{[2]} l_2^1 + i W^{[2]} l_2^2, \quad (3.61)$$

$$\lambda^2 : l_{2t}^2 - 2i l_2^0 S + l_2^1 S S_x - \frac{i}{\omega} l_2^2 W = -2i S^{[2]} l_2^0 + S^{[2]} S_x^{[2]} l_2^1 - \frac{i}{\omega} W^{[2]} l_2^2, \quad (3.62)$$

$$\lambda^3 : -2i l_2^1 S + l_2^2 S S_x = -2i S^{[2]} l_2^1 + S^{[2]} S_x^{[2]} l_2^2, \quad (3.63)$$

$$\lambda^4 : -2i l_2^2 S = -2i S^{[2]} l_2^2, \quad (3.64)$$

$$(\lambda + \omega)^{-1} : i l_2^0 W - i \omega l_2^1 W + i \omega^2 l_2^2 W = i W^{[2]} l_2^0 - i \omega W^{[2]} l_2^1 - i \omega^2 W^{[2]} l_2^2. \quad (3.65)$$

Hence we obtain the following 2-fold DT:

$$S^{[2]} = l_2^2 S (l_2^2)^{-1}, \quad (3.66)$$

$$W^{[2]} = (l_2^0 - \omega l_2^1 + \omega^2 l_2^2) W (l_2^0 - \omega l_2^1 + \omega^2 l_2^2)^{-1} \quad (3.67)$$

and

$$l_{2t}^0 = 0. \quad (3.68)$$

Note these equations we can rewrite as

$$S^{[2]} = L_{2\lambda\lambda} S (L_{2\lambda\lambda})^{-1}, \quad (3.69)$$

$$W^{[2]} = L_2|_{\lambda=-\omega} W (L_2|_{\lambda=-\omega})^{-1} \quad (3.70)$$

and

$$L_{2t}|_{\lambda=0} = 0, \quad (3.71)$$

respectively.

3.3 n-fold DT

Let us now we construct the n-fold DT. It has the form

$$\Phi^{[n]} = L_n \Phi^{[n-1]} = (\lambda N_n - I) \Phi^{[n-1]} = (\lambda N_n - I) \dots (\lambda N_2 - I) (\lambda N_1 - I) \Phi \quad (3.72)$$

so that

$$\Phi^{[n]} = [\lambda^n l_n^n + \lambda^{n-1} l_n^{n-1} + \dots + \lambda l_n^1 + l_n^0] \Phi, \quad (3.73)$$

where $l_n^0 = (-1)^n I$. The n -fold DT of the M-XCIX equation can be written as:

$$L_{nx} = U^{[n]} L_n - L_n U, \quad (3.74)$$

$$L_{nt} = V^{[n]} L_n - L_n V. \quad (3.75)$$

Hence, in particular, we obtain

$$S^{[n]} = \frac{\partial^n L_n}{\partial \lambda^n} S \left(\frac{\partial^n L_n}{\partial \lambda^n} \right)^{-1}, \quad (3.76)$$

$$W^{[n]} = L_n|_{\lambda=-\omega} W (L_n|_{\lambda=-\omega})^{-1} \quad (3.77)$$

and

$$L_{nt}|_{\lambda=0} = 0. \quad (3.78)$$

4 Soliton solutions

Now we consider a seed solution

$$S = \sigma_3, \quad W = b\sigma_3, \quad (4.1)$$

where $b = \text{const}(t)$. Then we get

$$S^{[1]} = \frac{1}{n} \begin{pmatrix} |n_{11}|^2 - |n_{12}|^2 & -2n_{11}n_{12} \\ -2n_{11}^*n_{12}^* & |n_{12}|^2 - |n_{11}|^2 \end{pmatrix}, \quad (4.2)$$

$$W^{[1]} = \frac{1}{\square} \begin{pmatrix} b(\omega^2 |n_{11}|^2 + \omega(n_{11} + n_{11}^*) + 1 - |n_{12}|^2) & -2\omega n_{11}n_{12} - 2n_{12} \\ -2\omega n_{11}^*n_{12}^* - 2n_{12}^* & -b(\omega^2 |n_{11}|^2 + \omega(n_{11} + n_{11}^*) + 1 - |n_{12}|^2) \end{pmatrix} \quad (4.3)$$

Now we are ready to write the solutions of the M-XCIX equation in terms of the elements of N . We get

$$S^{+[1]} = -\frac{2n_{11}^*n_{12}^*}{n}, \quad (4.4)$$

$$S^{-[1]} = -\frac{2n_{11}n_{12}}{n}, \quad (4.5)$$

$$S_3^{[1]} = \frac{|n_{11}|^2 - |n_{12}|^2}{n}, \quad (4.6)$$

$$W^{+[1]} = \frac{-2bn_{12}^*(\omega n_{11}^* + 1)}{\square}, \quad (4.7)$$

$$W^{-[1]} = \frac{-2bn_{12}(\omega n_{11} + 1)}{\square}, \quad (4.8)$$

$$W_3^{[1]} = \frac{(\omega^2|n_{11}|^2 + \omega(n_{11} + n_{11}^*) + 1 - |n_{12}|^2)b}{\square}. \quad (4.9)$$

In our case the eigenfunctions are given by

$$\psi_1 = e^{-i\lambda x + i[-2\lambda^2 + b(\frac{1}{\lambda+\omega} - \frac{1}{\omega})]t + i\delta_1} = e^{\theta_1 + i\chi_1}, \quad (4.10)$$

$$\psi_2 = e^{i\lambda x + i[2\lambda^2 - b(\frac{1}{\lambda+\omega} - \frac{1}{\omega})]t + i\delta_2} = e^{\theta_2 + i\chi_2}, \quad (4.11)$$

where

$$\theta_1 = \beta x - 2\alpha\beta t + \frac{\beta b}{(\alpha + \omega)^2 + \beta^2}t - \sigma_1, \quad (4.12)$$

$$\theta_2 = -\beta x - 2\alpha\beta t - \frac{\beta b}{(\alpha + \omega)^2 + \beta^2}t - \sigma_2, \quad (4.13)$$

$$\chi_1 = -i\alpha x - 2i\alpha^2 t - i\beta^2 t + \frac{ib(\alpha + \omega)}{(\alpha + \omega)^2 + \beta^2}t - \frac{i}{\omega}bt + i\tau_1, \quad (4.14)$$

$$\chi_2 = i\alpha x + 2i\alpha^2 t - i\beta^2 t - \frac{ib(\alpha + \omega)}{(\alpha + \omega)^2 + \beta^2}t + \frac{i}{\omega}bt + i\tau_2. \quad (4.15)$$

where $\lambda = \alpha + i\beta$ and $\delta_i = \tau_i + i\sigma_i$. Here $\alpha, \beta, b, \tau, \sigma, \delta_0$ are real constants. In this case the elements of N have the form

$$n_{11} = \frac{1}{\alpha^2 + \beta^2}(\alpha - i\beta \tanh 2\theta_1), \quad (4.16)$$

$$n_{12} = \frac{-i\beta e^{2i\chi_1 - i\delta_0}}{(\alpha^2 + \beta^2) \cosh 2\theta_1}, \quad (4.17)$$

and

$$\square = \omega^2 + 2\alpha\omega + \alpha^2 + \beta^2 = (\omega + \alpha)^2 + \beta^2, \quad (4.18)$$

Finally the 1-soliton solution of the M-XCIX equation has the form

$$S_3^{[1]} = \tanh^2 2\theta_1 + \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \frac{1}{\cosh^2 2\theta_1} = \frac{1}{\cosh^2 2\theta_1} \left(\sinh^2 2\theta_1 + \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \right), \quad (4.19)$$

$$S^{+[1]} = \frac{2\beta}{\alpha^2 + \beta^2} e^{-i\chi_1 + i\chi_2} \left(\frac{\beta \sinh 2\theta_1}{\cosh^2 2\theta_1} - \frac{i\alpha}{\cosh 2\theta_1} \right), \quad (4.20)$$

$$S^{-[1]} = \frac{2\beta}{\alpha^2 + \beta^2} e^{i\chi_1 - i\chi_2} \left(\frac{\beta \sinh 2\theta_1}{\cosh^2 2\theta_1} + \frac{i\alpha}{\cosh 2\theta_1} \right), \quad (4.21)$$

Similarly, we can find solutions for W . They have the form

$$W_3^{[1]} = \frac{b}{M} \left[\frac{\omega^2}{\cosh^2 2\theta_1} \left(\sinh^2 2\theta_1 + \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \right) + 2\alpha\omega + \frac{1}{\alpha^2 + \beta^2} \right], \quad (4.22)$$

$$W^{+[1]} = \frac{2\beta\omega e^{-2i\chi_1 + i\delta_0} b}{(\alpha^2 + \beta^2) M \cosh 2\theta_1} [\beta\omega \tanh 2\theta_1 - i(\alpha\omega + \alpha^2 + \beta^2)], \quad (4.23)$$

$$W^{-[1]} = \frac{2\beta\omega e^{2i\chi_1 - i\delta_0} b}{(\alpha^2 + \beta^2) M \cosh 2\theta_1} [\beta\omega \tanh 2\theta_1 + i(\alpha\omega + \alpha^2 + \beta^2)], \quad (4.24)$$

where

$$M = (\omega + \alpha)^2 + \beta^2. \quad (4.25)$$

Also we note that take place the following formula (which may be will useful to understand the nature of the matrix function W):

$$W^{[1]} = \frac{\omega^2 b}{M} S^{[1]} + F^{[1]}, \quad (4.26)$$

where

$$F_3^{[1]} = \frac{b}{M} \left(2\alpha\omega + \frac{1}{\alpha^2 + \beta^2} \right), \quad (4.27)$$

$$F^{+[1]} = -\frac{2i\beta\omega b e^{-2i\chi_1 - i\delta_0}}{M \cosh 2\theta_1}, \quad (4.28)$$

$$F^{-[1]} = \frac{2i\beta\omega b e^{2i\chi_1 + i\delta_0}}{M \cosh 2\theta_1}. \quad (4.29)$$

5 Conclusion

The Heisenberg ferromagnetic equations are fascinating nonlinear dynamical models. In particular, integrable Heisenberg ferromagnetic equations have much relevance in applied ferromagnetism and nanomagnetism. Integrable Heisenberg ferromagnetic equations have a close connection to the nonlinear Schrodinger models as well as to the differential geometry of curves and surfaces (see e.g. [24] and references therein). So it is considered to be one of the fundamental integrable equations admitting an N -soliton solution.

In soliton theory, the construction of the analytical and simple form of the solutions for the integrable partial and ordinary differential equations is one of central topics in recent years. In this context, up to now the Darboux transformation method, originated from the work of Darboux in 1882 for the Sturm-Liouville equation, is a powerful method to construct exact solutions for integrable linear and nonlinear differential equations. Various approaches have been proposed to find a Darboux transformation for a given integrable system, for example, the gauge transformation method, the loop group transformation, the operator factorization method. Indeed, through iterations, the Darboux transformation is often led to compact representations in terms of special determinants like as Grammian or Wronskian for the exact N -soliton solutions of the given equation. Such N -soliton solutions are appealing both from practical application viewpoint and from the theoretical viewpoint as well.

Since integrable systems have remarkable mathematical properties and numerous physical applications, their generalizations or extensions have attracted attention of many researchers. One possible direction is extensions with self-consistent sources. This sort of extensions may also be physically interested. The most famous example might be the nonlinear Schrodinger - Maxwell - Bloch equation, which now is one of the most important equations in theory of pulse propagation along the optical fiber. Another interesting fundamental soliton equation is the Heisenberg ferromagnetic equation with self-consistent sources [23]-[24].

In this paper, we have derived the DT of the M-XCIX equation including one-fold, two-fold and n -fold DT. This equation describe a nonlinear dynamics of the (1+1)-dimensional ferromagnets with self-consistent potentials and is integrable. As an example, the 1-soliton solution of the M-XCIX equation have been constructed explicitly by using the DT from some trivial seed solution. Of course, in a recursive manner, we can construct the N -soliton solution as well as the other type nonlinear solutions like: breathers, positons and so on. Moreover, this constructed Darboux transformation, in particular, allows us to calculate higher order rogue wave solutions of the M-XCIX equation and other Heisenberg ferromagnetic equations with self-consistent sources. We will study these important solutions of the M-XCIX equation as well as its integrable generalizations in future.

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